$\lim_{m\to\infty} |\xi_j^{(n)} - \xi_j^{(m)}| < \epsilon \quad \forall\, n, m \geq N.$

$\Rightarrow |\xi_j^{n} - \xi_j| < \epsilon \quad \forall\, n \geq N$ — ③

$\because x_n = \langle \xi_j^{(n)} \rangle \in l^\infty$

$\therefore \exists$ a real number K_n s.t.

$|\xi_j^{(n)}| \leq K_n \quad \forall\, j \in \mathbb{N}$ — ④

Now, $|\xi_j| = |\xi_j - \xi_j^{(n)} + \xi_j^{(n)}|$

$\leq |\xi_j^{(n)} - \xi_j| + |\xi_j^{(n)}|$

$< \epsilon + K_n \quad \forall\, n \geq N$ & $\forall j$ {using ③ & ④}

$\Rightarrow |\xi_j| \leq \epsilon + K_n \quad \forall\, n \geq N$ & $\forall j$

$\therefore \langle \xi_j \rangle$ is a bdd sequence

$\Rightarrow x = \langle \xi_j \rangle \in l^\infty$

Also from ③, we obtain,

$d(x_n, x) = \sup_{1 \leq j < \infty} |\xi_j^{n} - \xi_j| < \epsilon \quad \forall\, n \geq N.$

$\Rightarrow x_n \to x$ & $x \in l^\infty$

$\Rightarrow l^\infty$ is complete

$\Rightarrow (l^\infty, \|\cdot\|_\infty)$ is a Banach space

(4) The sequence space C is complete.

We know. $C \subset l^\infty$ & l^∞ is complete.

To show:- C is complete.

We'll show that C is closed in l^∞.

We know that $C \subset \overline{C}$.

To show: $\overline{C} \subset C$.

Let $x = \langle \xi_j \rangle_{j \geq 1}^{\infty} \in \overline{C}$ be arbitrary

Since $x \in \overline{C}$

$\Rightarrow \exists$ a sequence $\langle x_n \rangle_{n \geq 1}^{\infty}$ in C s.t $x_n \to x$

where $x_n = \langle \xi_j^{(n)} \rangle$

$\Rightarrow$ for each $\epsilon > 0$, $\exists$ a +ve integer $N = N(\epsilon)$ s.t

$\|x_n - x\|_\infty < \frac{\epsilon}{3} \quad \forall\, n \geq N.$

$\Rightarrow d_\infty(x_n, x) < \frac{\epsilon}{3} \quad \forall\, n \geq N.$

$\Rightarrow \sup_{1 \leq j < \infty} |\xi_j^{(n)} - \xi_j| < \frac{\epsilon}{3} \quad \forall\, n \geq N$